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Note

## Cyclomatic numbers of planar graphs

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**Abstract**

For a given planar graph  $G$  with a set  $A$  of independent vertices, we provide a best-possible upper bound for the minimum cyclomatic number of connected induced subgraphs of  $G$  containing  $A$ . The extremal graphs are also characterized. © 1998 Elsevier Science B.V.

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**1. Introduction**

This paper considers only simple, finite, planar graphs. Let  $G$  be a graph. If  $S \subseteq V(G)$ , then  $G[S]$  is the induced subgraph of  $G$ . For two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , we define  $G = G_1 \cap G_2$  to be the graph  $G = (V_1 \cap V_2, E_1 \cap E_2)$ . Define  $G = G_1 \cup G_2$  similarly.

Define a set  $A \subset V(G)$  to be *independent* in  $G$  if for all  $x, y \in A$ ,  $x$  is not adjacent to  $y$ . The *cyclomatic number* of a graph  $H$  is defined to be  $cy(H) = |E(H)| - |V(H)| + 1$ . Further, let  $H$  be an induced subgraph of  $G$ , and define the cyclomatic number of  $G$  with respect to an independent set  $A$  by  $cy(A, G) = \min\{cy(H) : A \subset H\}$ .

Motivated by Barnette's conjecture that every cubic 3-connected bipartite planar graph contains a Hamilton cycle, Alspach and Oral [1] posed the question: What can be said about  $cy(A, G)$ ? In [2], the author showed that  $cy(A, G) \leq \binom{|A|}{2}$  for general graphs and conjectured that  $cy(A, G) \leq 2|A| - 5$  for planar graphs. We will prove that this conjecture is true. The extremal graphs will also be characterized in certain cases. In order to prove our results, we need the following notation from [2]. Take a subset  $A$  of  $V(G)$  such that  $A$  is independent in  $G$ . Then we define  $C(A)$  to be the collection of all connected induced subgraphs of  $G$  containing  $A$ . If  $H \in C(A)$ , but any proper subgraph of  $H$  does not belong to  $C(A)$ , then  $H$  is said to be *minimal* in  $C(A)$ . We can also write  $cy(A, G) = \min\{cy(H) : H \text{ is minimal in } C(A)\}$ , where the minimum is taken among all connected induced subgraphs of  $G$  containing  $A$ .

We will consider only connected graphs, since if  $G$  is not connected and  $A$  is contained in more than one component of  $G$ , then  $C(A) = \emptyset$ .

## 2. Planar graphs

We first describe a class of graphs which will be used to characterize the extremal graphs. Let  $G$  be a planar graph,  $A$  a set of independent vertices in  $G$ , and  $H$  an induced subgraph of  $G$  such that  $H$  is minimal in  $C(A)$ . If there exists a subset  $M$  of  $H$  such that  $M$  induces a maximal planar subgraph, and such that  $|M \setminus A| = |A \setminus M|$  and  $H$  is obtained from  $M$  by adding disjoint paths from  $M \setminus A$  to  $A \setminus M$  (in particular, there is exactly one such path at each vertex of  $M \setminus A$ ; see Fig. 1), then we will say  $H \in \mathbb{H}(A, G)$ . We will refer to such paths as the *pendant paths* of  $H$ . For technical reasons, we will allow  $M = K_1$  and  $K_2$ .

**Theorem 2.1.** *Let  $G$  be a planar connected simple graph. Let  $A$  be an independent set of vertices in  $G$ , and let  $H$  be a subgraph of  $G$  such that  $H$  is minimal in  $C(A)$ . Then  $cy(H) \leq 2|A| - 5$  for  $|A| \geq 3$ , and  $cy(H) = 0$  if  $|A| \leq 2$ . Furthermore, equality holds if and only if  $H \in \mathbb{H}(A, G)$ .*

**Proof.** We will use induction on the number of vertices in  $H$ . It is easy to verify that  $cy(H) = 0$  if  $|A| \leq 2$ . Hence let  $|A| \geq 3$ .

*Claim 1.* We may assume no vertex in  $A$  is a cut-vertex of  $H$ .

Suppose that there exists a vertex  $v \in A$  such that  $v$  is a cut-vertex of  $H$ . Then denote the components of  $H \setminus \{v\}$  by  $H_1, H_2, \dots, H_k$ , where  $k \geq 2$ . Let  $m \leq k$  be the number

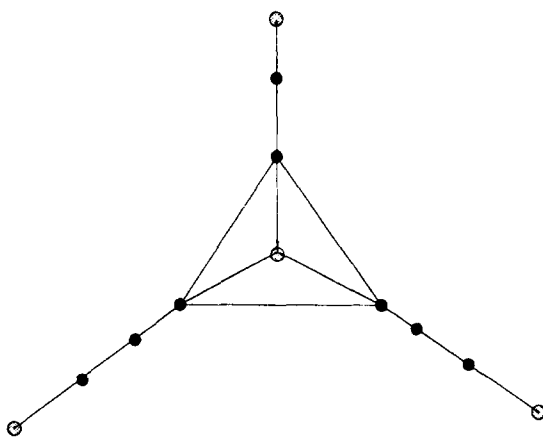


Fig. 1. An extremal graph where  $A$  is the set of shaded vertices.

of components of  $H \setminus \{v\}$  with  $|A_i| = |H_i \cap A| = 1$  (note that  $|H_i \cap A| \neq 0$ ; otherwise,  $H$  would not be minimal in  $C(A)$ ).

For each  $H_i$ ,  $H'_i = G[V(H_i) \cup \{v\}]$  is minimal in  $C(A_i \cup \{v\})$ ; otherwise,  $H$  would not be minimal in  $C(A)$ . Let  $A'_i = A_i \cup \{v\}$ . Then by induction,

$$cy(H'_i) \leq 2|A'_i| - 5 = 2(|A_i| + 1) - 5$$

for all  $H_i$  with  $|A'_i| \geq 3$ , and  $cy(H'_i) = 0$  if  $|A'_i| = 2$  (when  $|A_i| = 1$ ). By renumbering the components, suppose the first  $k - m$  components have  $|A_i| \geq 2$ . Now as  $|A| = (\sum_{i=1}^{k-m} |A_i|) + 1 + m$ , we have that

$$\begin{aligned} \sum_{i=1}^{k-m} cy(H'_i) &\leq \sum_{i=1}^{k-m} (2(|A_i| + 1) - 5) = \sum_{i=1}^{k-m} (2|A_i| - 3) \\ &= 2|A| - 3(k - m) - 2(1 + m) = 2|A| - 3k + m - 2 \\ &\leq 2|A| - 2k - 2 \leq 2|A| - 6 < 2|A| - 5. \end{aligned}$$

Hence

$$\begin{aligned} cy(H) &= \sum_{i=1}^k |E(H'_i)| - \sum_{i=1}^k |V(H'_i)| + k \\ &= \sum_{i=1}^k cy(H'_i) = \sum_{i=1}^{k-m} cy(H'_i) < 2|A| - 5. \end{aligned}$$

Thus, Claim 1 holds. Note that in the above case,  $cy(H) < 2|A| - 5$ ; that is, equality does not occur.

*Claim 2.* We may assume that for each  $v \in H \setminus A$ ,  $H \setminus v$  has exactly two components, one of which is a path from the unique neighbor of  $v$  in this component to a vertex in  $A$ .

Let  $v \in H \setminus A$ . Then since  $H$  is minimal in  $C(A)$ ,  $v$  is a cut-vertex of  $H$ . Write  $H = H_1 \cup H_2$ ,  $|V(H_i)| \geq 2$ , where  $H_1 \cap H_2 = \{v\}$ . Let  $H'_i$  be obtained from  $H_i$  by adding a vertex  $a_i$  and the edge  $a_i v$ , and let  $A'_i = A_i \cup \{a_i\}$  for  $i = 1, 2$ . Then for each  $i$ ,  $H'_i$  is minimal in  $C(A'_i)$ . Note that if  $|V(H'_i)| = |V(H)|$  for some  $i$ , then  $v$  satisfies Claim 2. So we may assume  $|V(H'_i)| < |V(H)|$  for  $i = 1, 2$ . If  $|A'_i| \geq 3$  (that is,  $|A_i| = |H_i \cap A| \geq 2$ ) for each  $i$ , then by induction

$$cy(H'_i) \leq 2|A'_i| - 5 = 2(|A_i| + 1) - 5 = 2|A_i| - 3.$$

Now

$$\sum_{i=1}^2 cy(H'_i) \leq 2|A| - 6.$$

Then we have that

$$cy(H) = \sum_{i=1}^2 cy(H'_i) < 2|A| - 5.$$

Therefore, without loss of generality, we may assume  $|A_1| = 1$ , and let  $A_1 = \{a\}$ . Then since  $H$  is minimal in  $C(A)$ ,  $H_1$  is a path between  $v$  and  $a$ , and Claim 2 follows.

By Claim 2, every  $v_i \in H \setminus A$  splits  $H$  into exactly two subgraphs, one of which is a path from  $v_i$  to a vertex  $a_i \in A$ . Among all such paths, the maximal ones are called *pendant paths*. Denote the vertices in  $H$  contained in such paths by  $N$ . Since  $|A| \geq 3$ ,  $N \neq V(H)$ . Let  $M$  be the induced subgraph of  $H$  such that  $V(M) = V(H) \setminus N$ . By Claim 2, we have that  $|A \setminus M| = |M \setminus A|$ . Now

$$\begin{aligned} cy(H) &= cy(M) = |E(M)| - |V(M)| + 1 \\ &\leq 3|V(M)| - 6 - |V(M)| + 1 = 2|V(M)| - 5. \end{aligned}$$

But by Claim 2 we have that

$$|V(M)| = |A \cap M| + |M \setminus A| = |A \cap M| + |A \setminus M| = |A|.$$

Therefore,

$$cy(H) = cy(M) \leq 2|A| - 5.$$

Note that in the above case,  $cy(M) = 2|A| - 5$  if and only if  $M$  has the maximal number of edges allowed for a planar graph,  $|M \setminus A| = |A \setminus M|$ , and every vertex  $v_i \in M \setminus A$  is connected by a pendant path to a corresponding vertex  $a_i \in A \setminus M$ . Hence,  $cy(H) = 2|A| - 5$  if and only if  $H \in \mathbb{H}(A, G)$ .  $\square$

**Corollary 2.2.** *For a given planar connected simple graph  $G$  with independent set  $A$ ,  $cy(A, G) \leq 2|A| - 5$  for  $|A| \geq 3$  and  $cy(A, G) = 0$  for  $|A| \leq 2$ .*

We now devote the remainder of this section to describing the structure of  $G$  when  $cy(A, G) = 2|A| - 5$ . Define an *H-bridge* of  $G$  to be either an edge in  $G \setminus H$  with both ends in  $H$ , or a component of  $G \setminus H$  together with the edges between  $H$  and that component. Those vertices in  $H$  which are contained in an *H-bridge* of  $G$  are called *attachments*. If  $P$  is a path and  $x, y \in P$ , then  $xPy$  will denote the section of  $P$  from  $x$  to  $y$  including  $x$  and  $y$ .

**Theorem 2.3.** *Let  $A$  be independent in  $G$  with  $|A| \geq 4$  and  $cy(A, G) = 2|A| - 5$ . Let  $H \in \mathbb{H}(A, G)$ , where  $H$  is obtained from a maximal planar graph  $M$  by adding pendant paths from  $M \setminus A$  to  $A \setminus M$ . Then for each *H-bridge*  $B$  of  $G$ ,  $B$  either has all attachments on a single pendant path of  $G$  or all attachments on a single facial cycle of  $M$ .*

**Proof.** Since  $cy(A, G) = 2|A| - 5$ , every subgraph of  $G$  which is minimal in  $C(A)$  is contained in  $\mathbb{H}(A, G)$ . We may assume  $H$  is embedded in the plane. A face of  $M$  is usually a connected component of  $\mathbb{R}^2 \setminus M$ , but here we interpret a face to be its closure in  $\mathbb{R}^2$ . Take an  $H$ -bridge  $B$  of  $G$ . Since  $G$  is planar, all attachments of  $B$  are in a common face of  $M$ . We may assume that  $B$  has all attachments in the infinite face of  $M$ . Denote the vertices in the infinite face of  $M$  by  $v_1, v_2, v_3$ .

Since each face of  $M$  contains at most three vertices in  $M \setminus A$ , we have at most three pendant paths from  $\{v_1, v_2, v_3\}$  to vertices in  $A \setminus M$ . Denote the pendant path from  $v_i \in M$  to  $a_i \in A \setminus M$  by  $P_i$ ,  $1 \leq i \leq 3$  ( $v_i, a_i \in P_i$ ). Then every attachment of  $B$  is contained in some  $P_i$ ,  $1 \leq i \leq 3$ . Suppose  $B$  has attachments on at least two of these paths. We show then that all attachments of  $B$  must be contained in  $\{v_1, v_2, v_3\}$ . Note that if a maximal planar graph contains at least four vertices, then it is 3-connected.

*Case 1:*  $B$  has attachments on exactly two paths, say  $P_1$  and  $P_2$ .

Suppose that  $B$  has an attachment  $u_1$  on  $P_1$  such that  $u_1 \neq v_1$  and  $u_1$  is closest to  $a_1$  on  $P_1$  ( $u_1 = a_1$  is possible). Denote by  $Q$  a shortest path from  $u_1$  to  $P_2$  through  $B$  such that every vertex of  $Q$  is in  $B$  except its ends, and let  $b_1$  be the last vertex in  $B \cap Q$  before reaching  $P_2$ . Let  $H'$  be the subgraph of  $G$  induced by vertices in  $[(H \setminus (v_1 P_1 u_1 \setminus u_1)) \cup Q]$ . By the choice of  $Q$ ,  $H'$  is minimal in  $C(A)$ . Now since  $\deg_H(v_1) \geq 3$ , we must have  $\deg_{H'}(b_1) \geq 3$ ; otherwise,  $cy(H') < cy(H)$ , a contradiction. Thus,  $b_1$  must be contained in the maximal planar part  $M'$  of  $H'$ . By the planarity of  $G$ ,  $\{v_2, v_3\}$  is a 2-cut of  $M'$ . Since  $|A| \geq 4$ , the maximal planar part of  $H'$  is 3-connected, a contradiction. Therefore,  $u_1 = v_1$ . Similarly, we can prove that  $v_2$  is the only attachment of  $B$  on  $P_2$ .

*Case 2:*  $B$  has attachments on all three paths  $P_i$ ,  $i = 1, 2, 3$ .

As in Case 1, let  $u_1 \neq v_1$  be the closest vertex on  $P_1$  to  $a_1$  attached to  $B$ . Assume first that  $B$  has an attachment  $u_3 \neq v_3$  on  $P_3$  such that  $u_3 P_3 a_3$  is shortest. Let  $Q_1$  be a path from  $u_1$  to  $P_2$  and  $Q_2$  a path from  $u_3$  to  $Q_1$  such that every vertex of  $Q_1$  and  $Q_2$  except possibly their ends is in  $B$ . We select  $Q_1$  and  $Q_2$  so that  $Q_1$  is shortest and subject to this,  $Q_2$  is shortest. Suppose that  $Q_1 \cap Q_2 = \{b\}$ , where  $b \in B$ . Now let  $H'$  be the graph induced by vertices in  $[(H \setminus \{(v_1 P_1 u_1 \setminus u_1) \cup (v_3 P_3 u_3 \setminus u_3)\}) \cup (Q_1 \cup Q_2)]$  (note that  $\deg_{H'}(b) \geq 3$ ). By the construction of  $Q_1$  and  $Q_2$ , it is easy to verify that  $H'$  is minimal in  $C(A)$ . By the planarity of  $G$ ,  $v_2$  is a cut-vertex of the maximal planar part  $M'$  of  $H'$ . Since  $|A| \geq 4$ , the above is a contradiction. Thus,  $u_3 = v_3$ . Similarly, we may assume  $B$  has no attachments on  $P_2$  other than  $v_2$ .

Now let  $Q$  be a shortest path from  $u_1$  to  $v_2$  such that  $Q \setminus \{u_1, v_2\} \subset B$ . Then  $G[(H \setminus (v_1 P_1 u_1 \setminus u_1)) \cup Q]$  contains an induced subgraph  $H'$  which is minimal in  $C(A)$ . Clearly,  $M \setminus v_1$  is contained in the maximal planar part  $M'$  of  $H'$ . Since  $M'$  is maximal planar,  $M' \neq M \setminus v_1$ . But by the planarity of  $G$ ,  $\{v_2, v_3\}$  is a 2-cut of  $M'$ , a contradiction. Hence all attachments of  $B$  are contained in  $\{v_1, v_2, v_3\}$ .  $\square$

It is not difficult to verify that the converse of Theorem 2.3 is also true for  $|A| \geq 4$ . Thus, we have characterized graphs for which  $cy(A, G) = 2|A| - 5$  when  $|A| \geq 4$ :  $G$  is obtained from an  $H \in \mathbb{H}(A, G)$  by adding  $H$ -bridges which either have all attachments

on a single pendant path or all attachments in a face of  $M$ . The remaining case, where  $|A| = 3$ , reduces to finding  $G$  such that  $G$  contains no induced subtree containing  $A$ . This is a special case of the following: characterize  $G$  with an induced tree containing a given independent set. We believe it could be a difficult problem.

### 3. Bipartite planar graphs

Let  $G$  be a bipartite planar graph and let  $A$  be an independent set of vertices in  $G$ . Define  $\mathcal{H}(A, G)$  in the same manner as we defined  $\mathbb{H}(A, G)$  above, with the exception that  $M$  is a maximal bipartite planar graph.

**Theorem 3.1.** *Let  $G$  be a planar connected simple bipartite graph. Let  $A$  be an independent set of vertices in  $G$ , and let  $H \subset G$  be minimal in  $C(A)$ . Then  $cy(H) \leq |A| - 3$  for  $|A| \geq 3$ , and  $cy(H) = 0$  for  $|A| \leq 2$ . Moreover, equality holds if and only if  $H \in \mathcal{H}(A, G)$ .*

**Proof.** The proof follows that of Theorem 2.1. There are only two minor differences in the proof. First, in Claim 1, we can get equality when  $k = m = 2$ . In this case,  $|A| = 3$  and the graph  $H$  is a path containing the three vertices in  $A$ , so  $cy(H) = 0 = |A| - 3$ . And second, at the end of the proof, we use the fact that since  $H$  is bipartite,  $H$  is triangle-free, and thus  $|E(H)| \leq 2|V(H)| - 4$ .  $\square$

**Corollary 3.2.** *For a given planar connected simple bipartite graph  $G$  with independent set  $A$ ,  $cy(A, G) \leq |A| - 3$  for  $|A| \geq 3$  and  $cy(A, G) = 0$  for  $|A| \leq 2$ .*

Some extremal graphs  $G$ , where every  $H \subset G$  such that  $H$  is minimal in  $C(A)$  has  $cy(H) = |A| - 3$ , can also be described as in Section 2, letting  $A$  be an independent set of vertices of  $G$  and  $M$  a maximal bipartite planar subgraph of  $H$ . The proof differs slightly, as every face of  $M$  is a quadrangle since  $M$  is a maximal bipartite graph. Thus, every  $H$ -bridge of  $G$  with no attachments in  $H \setminus M$  can have at most four attachments in  $M$ . In addition, each bridge must be constructed so that  $G$  remains a bipartite graph.

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